Time Delay Interferometry and LISA Optimal Sensitivity

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Abstract

The sensitivity of LISA depends on the suppression of several noise sources; dominant one is laser frequency noise. It has been shown that the six Doppler data streams obtained from three space-crafts can be appropriately time delayed and optimally combined to cancel this laser frequency noise. We show that the optimal data combinations when operated in a *network* mode improves the sensitivity over Michelson ranging from 40% to 100%. In this article, we summarize these results. We further show that the residual laser noise in the optimal data combination due to typical arm-length inaccuracy of 200 m is much below the level of optical path and the proof mass noises.

1 Introduction

The future space-based gravitational wave (GW) mission [1]- the Laser Interferometric Space Antenna (LISA) - consists of three identical space-crafts forming an equilateral triangle of side 5×10^6 km following heliocentric orbits trailing the Earth by 20°. LISA is thus a giant interferometric configuration with three arms which will give independent information on GW polarizations and will detect GW in the low frequency range of 0.1 mHz to 0.1 Hz. Due to the long arm-lengths of the antenna, it is not feasible to bounce the laser beams. A special Doppler tracking scheme is used to track the space-crafts with laser beams. This exchange of laser beams between the three space-crafts result in six Doppler data streams. The LISA sensitivity is limited by many noise sources; the dominant one is the laser phase noise; noise due to phase fluctuations of the master laser. The current stabilization schemes estimate this noise to about $\Delta\nu/\nu_0 \simeq 10^{-13}/\sqrt{Hz}$, where ν_0 is the frequency of the laser and $\Delta\nu$ the fluctuation in frequency. If the laser frequency noise can be suppressed then the noise floor is determined by the optical-path noise which fakes the fluctuations in the lengths of optical paths and the residual acceleration of proof masses resulting from imperfect shielding of the drag-free system. Thus, canceling the laser frequency noise is vital for LISA

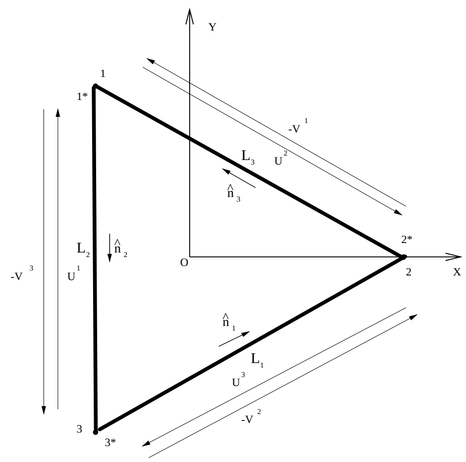

Figure 1: The LISA constellation

to reach the requisite sensitivity of $h \sim 10^{-21}$ or 10^{-22}. Since it is impossible to maintain equal distances between space-crafts, cancellation of laser frequency noise is a non-trivial problem. Several schemes have been proposed to combat this noise. In these schemes [2,3], the data streams are combined with appropriate time delays in order to cancel the laser frequency noise. In our earlier work, it was rigorously shown that *all* laser-noise free data combinations form an algebraic module over a polynomial ring over time delay operators [4]. Furthermore, recently [5], we have found data combinations which are eigen-data combinations of the noise covariance matrix, which give optimal sensitivity, when averaged over all the source directions and polarization angles. We show that signal-to-noise ratio (SNR) for any data combination in the laser noise-free cancellation module lies between the SNR's of these eigen-data combinations. In this article, we summarize our previous results [4,5]. Besides, we estimate the residual laser noise due to inaccuracies in the arm-lengths.

2 Time Delay Interferometry and Laser Noise Cancellation Module

In LISA constellation, the six Doppler data streams are labeled as U^i and $V^i, i = 1, 2, 3$; if the space-crafts are labeled clockwise as shown in Fig. 1. The Doppler data stream U^1 is obtained by letting the laser beam from space-craft 3 to travel towards space-craft 1 along the arm of length L_2 in the direction $-\hat{\mathbf{n}}_2$, and is beaten with the on-board laser beam at space-craft 1. Similarly, $-V^1$ represents the Doppler data obtained by beating laser beam traveling from space-craft 2 to space-craft 1 along the arm of length L_3 in the direction of $\hat{\mathbf{n}}_3$ with the on-board laser at space-craft 1. The remaining 4 beams are described by cyclically permuting the indices. These beams contain the laser frequency noise, other noises such as optical path, acceleration etc. and also the GW signal. Using time delay interferometry, these 6 Doppler data streams can be appropriately combined by delaying them with time delay operators $E_i x(t) = x(t - L_i), i = 1, 2, 3$ such that the resultant data streams is laser noise-free. Any laser noise-free data combination A is represented by

$$A = p_i V^i + q_i U^i \,, \tag{1}$$

where p_i, q_i are polynomials of time-delay operators E_i. For the LISA configuration $L_i \sim 16.7$ seconds, corresponding to an arm-length of 5 million km, one such known laser noise-free data combination, expressed in terms of 6-tuples of polynomials $(p_i,\, q_i)$ is

$$X = (1 - E_2^2, 0, E_2(E_3^2 - 1), 1 - E_3^2, E_3(E_2^2 - 1), 0) \tag{2}$$

commonly referred to as *Michelson* combination in the literature [6]; in which one arm of LISA is not used. In general, we have shown [4] that *all* the data combinations which cancel

the laser frequency noise and the optical bench motion noise form an algebraic module of *syzygies* over a ring of polynomials in time delay operators E_i. This formalism generates the noise cancellation module from the generators; linear combinations of the generators with polynomial coefficients in the ring generates a module. One such set of generators (which is convenient for SNR optimization purpose) is α, β, γ and ζ (notation followed from [2,3,4]).

$$\begin{aligned}
\alpha &= (1, E_3, E_1E_3, 1, E_1E_2, E_2)\,, \\
\beta &= (E_1E_2, 1, E_1, E_3, 1, E_2E_3)\,, \\
\gamma &= (E_2, E_2E_3, 1, E_3E_1, E_1, 1)\,, \\
\zeta &= (E_1, E_2, E_3, E_1, E_2, E_3)\,.
\end{aligned} \tag{3}$$

We note that α, β, γ are cyclic permutations of each other. The combination ζ is the symmetric Sagnac combination which is insensitive to GW at low frequency due to its high symmetry .

3 LISA Sensitivity Optimization

To generate any laser noise-free data combination, in general, 4 generators are necessary but if the source is monochromatic, the fourth generator ζ can be effectively eliminated by expressing in terms of (α, β, γ) as follows

$$(1 - E_1E_2E_3)\zeta = (E_1 - E_2E_3)\alpha + (E_2 - E_1E_3)\beta + (E_3 - E_1E_2)\gamma\,. \tag{4}$$

except at certain frequencies which are solutions of $e^{i(L_1+L_2+L_3)\Omega} = 1$. As the maximization is possible arbitrarily close to the singular frequencies, the singularities do not seem to be important.

Since the difference in arm-lengths of LISA is smaller than the GW wavelength [1], for computing the response, all the arms can be taken to be equal. This simplifies further analysis. We then show that the set of 3 optimal data combinations having noises uncorrelated to each other can be obtained by linearly combining α,β,γ. This new set acts as optimal when one averages over all directions and polarizations of the binary system. Below, we briefly summarize this optimization.

(a) Noise covariance matrix: We define noise vectors in the Fourier domain [4] $N^{(I)}$, $I = 1, 2, 3$ for each of the generators $X^{(I)} \equiv \{\alpha, \beta, \gamma\}$, respectively, over the 12 dimensional complex space $\mathcal{C}^{12}$,

$$N^{(I)} \;=\; \left(\sqrt{S^{pf}}(2p_i^{(I)} + r_i^{(I)}), \sqrt{S^{pf}}(2q_i^{(I)} + r_i^{(I)}), \sqrt{S^{sh}}p_i^{(I)}, \sqrt{S^{sh}}q_i^{(I)}\right)\,, \tag{5}$$

where $S^{pf}(f) = 2.5 \times 10^{-48}\,[f/1\,Hz]^{-2}\;Hz^{-1}$ and $S^{opt}(f) = 1.8 \times 10^{-37}\,[f/1\,Hz]^{2}\;Hz^{-1}$ are power spectral densities (psd) of the proof mass residual motion and the optical path noise respectively [3]. The polynomials $(p_i^{(I)}, q_i^{(I)})$ corresponding to the generators $X^{(I)}$ are given in the equation (3). The $r_i^{(I)}$ polynomials are defined through $r_1^{(I)} = -(p_1^{(I)} + E_3\, q_2^{(I)}) = -(q_1^{(I)} + E_2\; p_3^{(I)})$ plus cyclic permutations for $r_2^{(I)}$ and $r_3^{(I)}$. For a given data combination, the norm of the noise vector represents its noise psd. The noise covariance matrix $\mathcal{N}^{(I)}_{(J)} = N^{(I)} \cdot N^{*}_{(J)}$ defined for above generating set $X^{(I)}$ takes a symmetric form as

$$\mathcal{N}^{(I)}_{(J)} = \{n_d \;\text{ for } I = J \text{ and } n_o \;\text{ for } I \neq J\}. \tag{6}$$

(b) Signal covariance matrix: The response of a GW signal for a given laser noise-free data combination A is expressed in the Fourier domain [4] as,

$$h^{(A)}(\Omega) = \sum_{i=1}^{3} \left[p_i^{(A)}\left(F_{V_{i;+}}h_+ + F_{V_{i;\times}}h_\times\right) + q_i^{(A)}\left(F_{U_{i;+}}h_+ + F_{U_{i;\times}}h_\times\right)\right](\Omega)\,. \tag{7}$$

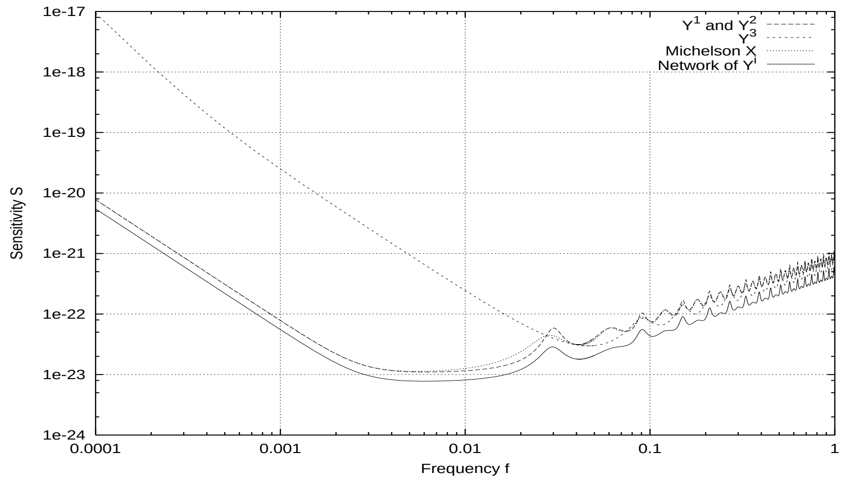

Figure 2: Log Log plot of sensitivity S, curve as function of f after averaging over polarization and source directions for a observation period of one year and SNR =5.

Here, $F_{V_{i;+/\times}}$ and $F_{U_{i;+/\times}}$ are the antenna pattern functions. For a binary source which may be adiabatically changing in frequency, the two GW amplitudes at frequency Ω are given by,

$$
\begin{aligned}
h_+(\Omega) &= \mathcal{A}\left(\frac{1+\cos^2\epsilon}{2}\cos 2\psi - i\cos\epsilon\, sin2\psi\right), \\
h_\times(\Omega) &= \mathcal{A}\left(-\frac{1+\cos^2\epsilon}{2}sin2\psi - i\cos\epsilon\,\cos 2\psi\right).
\end{aligned}
$$

Here, the polarization angles ϵ and ψ describe the orientation of the source and enter into the expressions for the polarization amplitudes. The direction of the source on the celestial sphere is given by the angles θ and ϕ. Similar to noise covariance matrix, the signal covariance matrix averaged over all directions and polarizations is defined as,

$$
\mathcal{H}^{(I)}_{(J)} = \langle h^{(I)} h^{*}_{(J)} \rangle_{\epsilon\psi\theta\phi}\,, \tag{8}
$$

where $\langle\ \rangle_{\epsilon\psi\theta\phi}$ represents the average over the polarizations and directions. We note that $\mathcal{H}^{(IJ)}$ takes the same symmetric structure as the noise covariance matrix $\mathcal{N}^{(IJ)}$ given in equation (6). This is due to inbuilt cyclic symmetry in this generating set $X^{(I)}$. Thus, the diagonal component of $\mathcal{H}^{(IJ)}$ is h_d and the off-diagonal is h_o.

Due to above symmetric structure, both $\mathcal{H}^{(IJ)}$ and $\mathcal{N}^{(IJ)}$ can be diagonalized simultaneously. The common eigen-observables thus obtained are given by

$$
Y^{(1)} = \frac{1}{\sqrt{6}}\left(\alpha+\beta-2\gamma\right)\,, \quad Y^{(2)} = \frac{1}{\sqrt{2}}\left(\beta-\alpha\right)\,, \quad Y^{(3)} = \frac{1}{\sqrt{3}}\left(\alpha+\beta+\gamma\right)\,. \tag{9}
$$

The above eigen-observables have following properties:

- Data combinations $Y^{(1)}$ and $Y^{(2)}$ have same SNR given by $\sqrt{(h_d-h_o)/(n_d-n_o)}$. The SNR of $Y^{(3)}$ is $\sqrt{(h_d+2h_o)/(n_d+2n_o)}$. At any given frequency, if SNR of $Y^{(1)}$ is greater than SNR of $Y^{(3)}$, then $Y^{(1)}$ gives maximimum SNR whereas $Y^{(3)}$ gives minimum SNR or *vice-a-versa*. Thus, at any given frequency, either $Y^{(1)}$ or $Y^{(3)}$ acts as optimal data combination amongst all the data combinations in the laser noise-free module.
- Both $Y^{(1)}$ and $Y^{(2)}$ perform comparable to the Michelson combination X at low frequency. Whereas $Y^{(3)}$ is proportional to the symmetric Sagnac and hence insensitive to GW at low frequencies ($\sim$ below 3 mHz).

We note that averaging over directions and polarizations[5] results in signal covariance matrix of rank 3. We note that if we average over the polarizations only and obtain the optimal

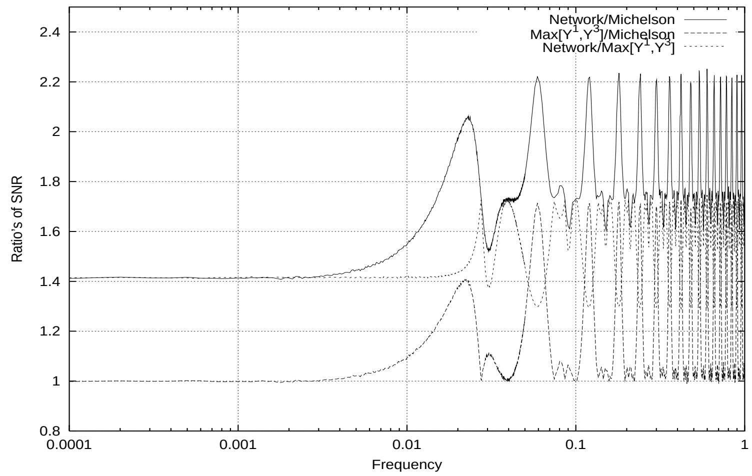

Figure 3: Plots showing the relative improvements (ratios) of SNRs for the three cases: (i) Network SNR over the Michelson data combination (solid line). (ii) Network SNR over the maximum of Max $[Y^{(1)}, Y^{(3)}]$ (dotted line). (iii) Max $[Y^{(1)}, Y^{(3)}]$ over the Michelson (dashed line). Here Max $[Y^{(1)}, Y^{(3)}]$ is the maximum of the SNR of $Y^{(1)}$ and $Y^{(3)}$ over the bandwidth of LISA.

data combination for a particular direction, the signal covariance matrix is of rank 2. Such an optimal combination is important while optimally tracking the source in LISA frame [7]. We may further note that in the formalism developed by Prince *et al* [8], the optimization is performed without averaging over the source directions and polarizations, which results in signal covariance matrix of rank 1.

We have shown that either $Y^{(1)}, Y^{(2)}$ or $Y^{(3)}$ maximize the LISA sensitivity on an average sense and they are orthogonal *i.e.* they are independent random variables. The sensitivity of LISA can be further improved as each of these generators can be realized as independent gravitational wave detectors. We assume that the $Y^{(I)}$ follow the Gaussian noise distribution and thus quadratically can combine the SNR's of these eigen-observables to form a *network*-observable [9]. The network SNR is given by

$$SNR^2_{network} = \sum_{I=1}^{3} SNR^2_{(I)} = 2SNR^2_{Y(1)} + SNR^2_{Y(3)} \,. \tag{10}$$

The corresponding sensitivities are shown in the Fig. 2. In Fig. 3, we have plotted the relative improvements in the network SNR with respect to the Michelson combination and the maximum of $Y^{(1)}$ and $Y^{(3)}$. At low frequencies $f \lesssim 15$ mHz, the improvement of the network SNR over the maximum of $Y^{(I)}$ is slightly greater than $\sqrt{2}$. This is because at low frequencies the data combination $Y^{(3)}$ is not very sensitive in comparison with $Y^{(1)}$. The best improvement of factor $\sqrt{3}$ in the relative SNR is achieved at frequencies where all the data combinations are equally sensitive, that is, when $SNR_{Y(1)} = SNR_{Y(3)}$.

4 Residual laser noise

As we know, it is difficult to maintain constant distance between the three space-crafts. In general, the three arm-lengths will be different. The actual length L'_i can be estimated upto a certain accuracy. Let the estimated arm-length be L_i with an error ΔL_i such that the actual length is $L'_i = L_i + \Delta L_i$. Because of this unknown inaccuracy, the laser noise is not completely canceled in a given laser noise-free data stream. If we demand that this residual laser phase noise level should be below the combined noise from proof mass S_{pf} and the optical path noise S_{opt} then this puts an upper limit on the laser stabilization requirement $\widetilde{\Delta\nu}$ and is given by

$$\widetilde{\Delta\nu} = \frac{\nu_0}{\Omega \Delta L} \left[\frac{S_{pf} + S_{opt}}{\sum |p_i|^2 + |q_i|^2} \right]^{1/2} \,. \tag{11}$$

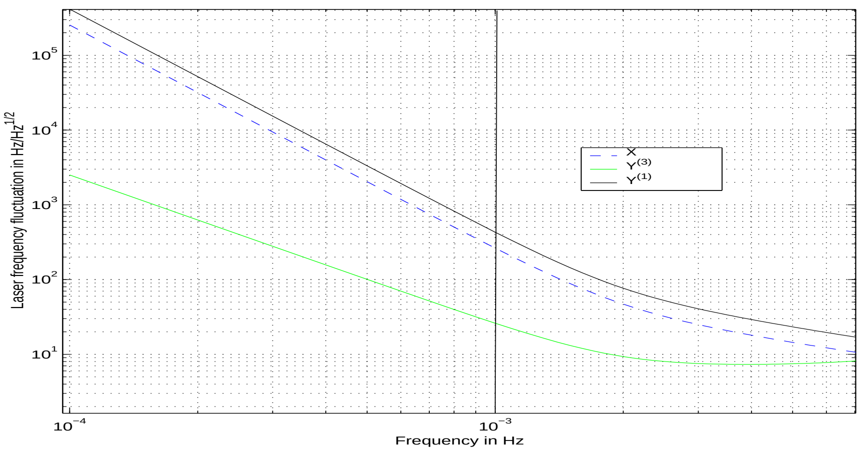

Figure 4: Laser frequency stabilization $\widetilde{\Delta\nu}$ in Hz/$\sqrt{\text{Hz}}$ as function of frequency for $Y^{(1)}$, $Y^{(3)}$ and Michelson X combinations, for $\Delta L = 200$ m.

For Nd-YAG laser, $\nu_0 = 3 \times 10^{14}$ Hz and the inaccuracy in length is assumed to be $\Delta L \sim 200$ m, the upper limit on the laser stabilization requirement is plotted for data combinations $Y_{(I)}$ in Fig. 4. For this assumed inaccuracy in arms, the $Y^{(3)}$ combination demands that the laser frequency stabilization be at least as good as $\widetilde{\Delta\nu} \sim 25$ Hz/$\sqrt{\text{Hz}}$ at 1 mHz. While for $Y^{(1)}$, the requirement is much less stringent on frequency stabilization. The laser stabilization requirement scales linearly with the assumed arm-length inaccuracy.

5 Conclusion

For any frequency in LISA band, we show that the optimal laser noise-free data combinations (when averaged over all directions and polarizations of GW source) are nothing but the eigen-data combinations of noise-covariance matrix. Since these combinations have uncorrelated noise, their SNR's can be combined quadratically to improve LISA sensitivity. The improvement varies from 40% to 100% with respect to Michelson data combination. We further show that with our demand that the residual laser noise should be less than the proof mass noise and optical path noise, the laser frequency stabilization requirement varies inversely proportional to the arm-length inaccuracy. The stringent demand on the laser stabilization requirement is $\widetilde{\Delta\nu} \sim 25$ Hz/$\sqrt{\text{Hz}}$ at 1 mHz for inaccuracies in arm-length of 200 m.

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